ELECTROMAGNETIC FIELD THEORY
Unit – I

Objectives:

- To introduce the concepts of Electrostatics and Magneto statics.

Syllabus:

Basic Concepts: Coulomb's Law, Electric field intensity, Electric fields due to Point Charge, line charge, surface charge and volume charge distributions, Electric Flux Density, Gauss's law, Applications of Gauss law: Point Charge, Infinite Line Charge.

Outcomes:

Students will be able to

- Understand basic law on point charges.
- Calculate the electric fields due to different charge distributions.
- Understand Gauss law and its applications.
- Gauss law on different charge configurations.

INTRODUCTION:

Electrostatics can be defined as the study of electric charges at rest. Electric fields have their sources in electric charges.(Note: Almost all real electric fields vary to some extent with time. However, for many problems, the field variation is slow and the field may be considered as static. For some other cases spatial distribution is nearly same as for the static case even though the actual field may vary with time. Such cases are termed as quasi-static.)

In this chapter we first study two fundamental laws governing the electrostatic fields, viz, (1) Coulomb's Law and (2) Gauss's Law. Both these law have experimental basis. Coulomb's law is applicable in finding electric field due to any charge distribution, Gauss's law is easier to use when the distribution is symmetrical.

COULOMB'S LAW:

Coulomb's Law states that the force between two point charges Q1and Q2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

$F = \frac{kQ_1Q_2}{R^2}$ Mathematically, , where k is the proportionality constant.

In SI units, Q1 and Q2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) $k = \frac{1}{4\pi\varepsilon_0}$ and , ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0\varepsilon_r$ instead ε_0 where ε_r is called the relative permittivity or the dielectric constant of the medium).

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$$ Therefore .... (1)

As shown in the Figure 1 let the position vectors of the point charges $Q_1$and Q_2 are given by $\mathbf{r_1}$ and $\mathbf{r_2}$.

Let $\mathbf{F_{12}}$ represent the force on Q1 due to charge Q2.

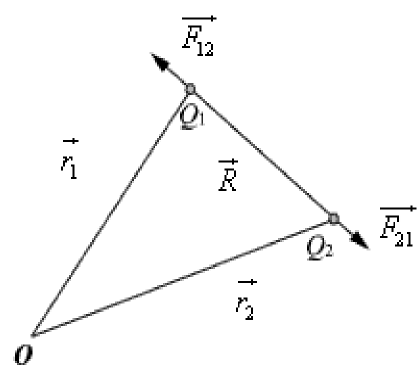


Fig. 1: Coulomb's Law

The charges are separated by a distance of $R = |\vec{r}_1 - \vec{r}_2| = |\vec{r}_2 - \vec{r}_1|$. We define the unit vectors as

$$\hat{a}_{21} = \frac{(\vec{r}_1 - \vec{r}_2)}{R} \qquad \hat{a}_{12} = \frac{(\vec{r}_2 - \vec{r}_1)}{R}$$

and can be defined as $\mathbf{F_{12}}$

$$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$ ....(2)

Similarly the force on Q_1 due to charge Q_2 can be $\vec{F}_{21}$ calculated and if represents this force then we can write $\vec{F}_{21} = -\vec{F}_{12}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have N number of charges $Q_1, Q_2, \ldots\ldots Q_N$ located respectively at the points represented by the position vectors $\vec{r}_1$, $\vec{r}_2$,...... $\vec{r}_N$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i(\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

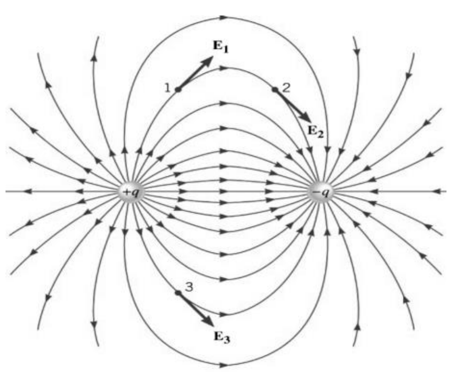
Limitations of Coulomb's Law:

- It is only applicable for point charges at rest.
- It is only applicable in dose cases where inverse square law is obeyed.
- It is difficult to apply the Coulomb's law when the charges are in arbitrary shape. Hence, we cannot determine the value of distance 'd' between the charges when they are in arbitrary shape.

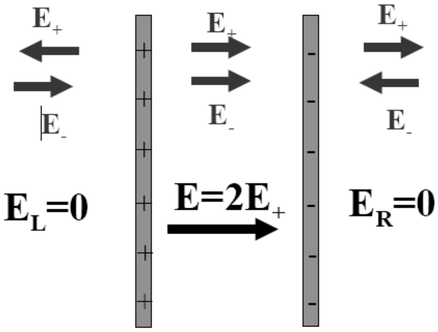
Applications of lines-of-force model

a) Dipole

The electric field line between two charges can be calculated using coulombs law of force.



b) Electric field between parallel plate capacitor .



ELECTRIC FIELD

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q} \quad \text{.......................................(4)}$$

The electric field intensity E at a point r (observation point) due a point charge Q located at $\vec{r}\,'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \quad \text{..(5)}$$

For a collection of N point charges Q_1 ,Q_2 ,.........Q_N located at **r1, r2.........rN** the electric field intensity at point $\vec{r}\,'$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3} \quad \text{..(6)}$$

The expression (6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2 we consider a continuous volume distribution of charge (t) in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r}\,')dv'$, i.e. considering this charge as point charge,

we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} = \frac{\rho(\vec{r}\,')dv'(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \quad \text{............(7)}$$

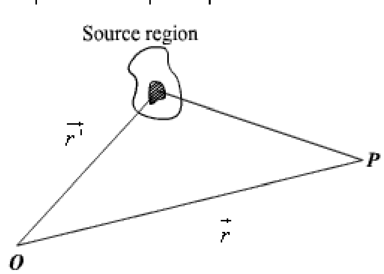


Fig. 2: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point P due to this distribution of charges. Thus the expression for the electric field at P can be written as:

$$\overrightarrow{E(r)} = \int_V \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dv'$$ ....(8)

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overrightarrow{E(r)} = \int_L \frac{\rho_L(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dl'$$ ....(9)

$$\overrightarrow{E(r)} = \int_S \frac{\rho_s(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} ds'$$ ....(10)

Electric Fields due to Point Charge

- The electric field intensity due to a point charge is given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R$$ ....(11)

Electric Fields due to Line Charge

- Consider a line charge with uniform charge density *pL* extending from *A* to *B* along the z-axis as shown in Figure. The charge element *dQ* associated with element *dl* = *dz* of the line is

$$dQ = \rho_L\, dl = \rho_L\, dz$$

- The total charge *Q* is

$$Q = \int_{z_A}^{z_B} \rho_L\, dz$$

- for a finite line charge

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o\rho}[-(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_z]$$

............................(12)

As a special case, for an infinite line charge, point B is at (0, 0, α) and A at (0, 0, -α), so that α1 = π/2, α2 = -π/2; the z-component vanishes and becomes

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$$ ....(13)

Electric Fields due to Surface Charge

- Consider an infinite sheet of charge in the xy-plane with uniform charge density *ps.* The charge associated with an elemental area *dS* and the total charge are

$$dQ = \rho s\ dS$$

$$Q = \int \rho_S\, dS$$

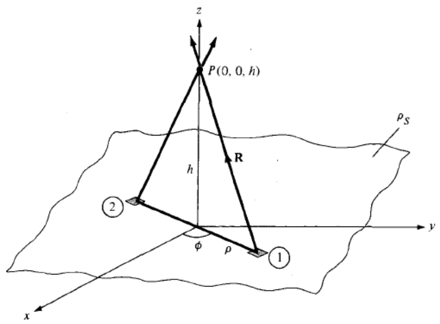


Fig.2a: Electric field intensity due to surface charge

- E has only z-component if the charge is in the xy-plane. Electric Fields due to Surface Charge is

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Electric Fields due to Volume Charge

- Let the volume charge distribution with uniform charge density *v.* The charge *dQ* associated with the elemental volume *dv* is

$$dQ = \rho v\ dv$$

-
- the total charge in a sphere of radius *a* is

$$Q = \int \rho_v\, dv = \rho_v \int dv = \rho_v \frac{4\pi a^3}{3}$$

- The electric field intensity due to a volume charge is given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

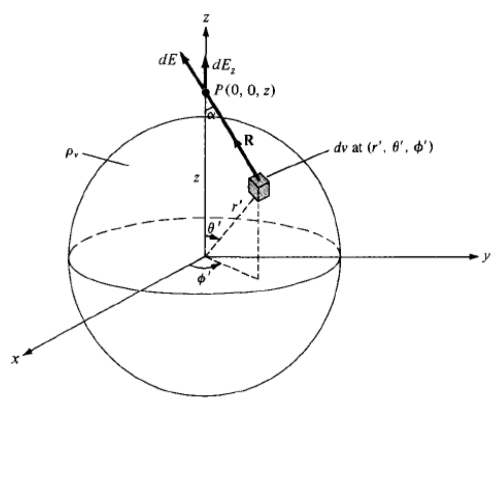


Fig. 2b: E due to volume charge

ELECTRIC FLUX DENSITY:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the

material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$\mathbf{D}= \varepsilon_0\mathbf{E}$

We define the electric flux as $\psi = \int_S \vec{D} . d\vec{s}$

Electric flux lines:

1. These are imaginary lines which shows pictorial influence of electric field intensity in space.
2. These are always emitted from positive charge and enter in negative charge.
3. They don't form closed loops.
4. These are always perpendicular to charge surface.
5. These lines never intersect with each other.

GAUSS'S LAW: Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

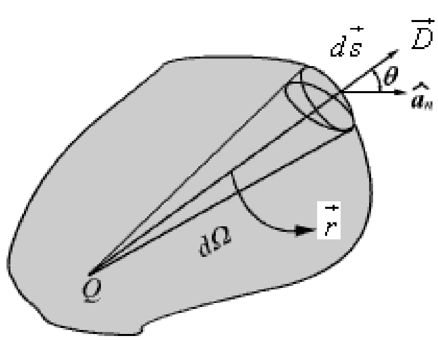


Fig. 3: Gauss's Law

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r$$ ....(18)

If we consider an elementary area ds, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2}ds\cos\theta$$ ....(19)

Therefore we can write $d\vec{s}$

$$d\psi = \frac{Q}{4\pi}d\Omega$$

For a closed surface enclosing the charge, we $\psi = \oint_S d\psi = \frac{Q}{4\pi}\oint_S d\Omega = Q$ can write

which can seen to be same as what we have stated in the definition of Gauss's Law.

APPLICATION OF GAUSS'S LAW:

Gauss's law is particularly $\vec{E}$ useful in computing or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density ρ_LC/m. Let us consider a line charge positioned along the z-axis as shown in Figure. Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Figure.

If we consider a close cylindrical surface as shown in Figure , using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0\vec{E}.d\vec{s} = \int_{S_1}\varepsilon_0\vec{E}.d\vec{s} + \int_{S_2}\varepsilon_0\vec{E}.d\vec{s} + \int_{S_3}\varepsilon_0\vec{E}.d\vec{s}$$ ....(20)

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero.

Hence we can write, $\rho_L l = \varepsilon_0 E.2\pi\rho l$

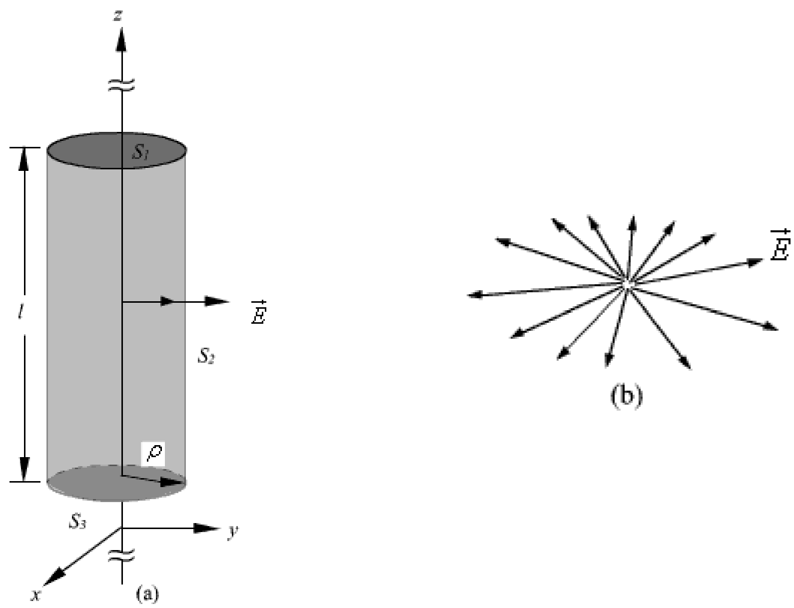


Fig 4: Infinite Line Charge

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$ ....(21)

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the x-z plane as shown in figure 5. Assuming a surface charge density of

ρ_s for the infinite surface charge, if we consider a cylindrical volume Δs having sides placed symmetrically as shown in figure 5, we can write:

$$\oint_S \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$ ....(22)

$$\therefore \qquad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

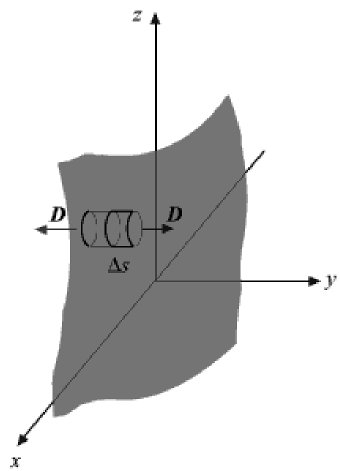


Fig. 5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

UNIT-2

Syllabus:

UNIT – II: Electrostatics-II

Energy expended in moving a point charge in an electric field, Electric Potential difference and Potential, Potential due to different charge configurations, Potential gradient, Electric dipole and Energy density in electrostatic field. Conduction and Convection Current, Current density.

Outcomes:

Students will be able to

- Understand the energy expended in moving a point charge in an electric field
- Calculate the electric potential between two points
- Differentiate conduction and convection currents.

Energy expended in moving a point charge in an electric field:

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field. Let us suppose that we wish to move a positive test charge from a point P to another point Q as shown in the Fig. 8.The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from P to Q. The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

$dW = -\Delta q \vec{E}.d\vec{l}$ .... (1)

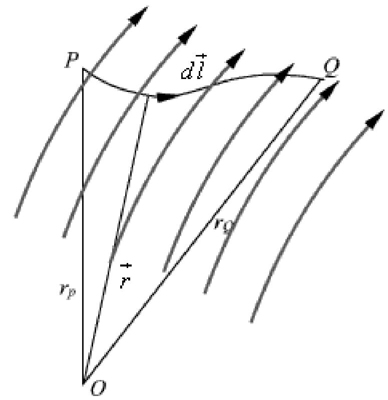


Fig.: Movement of Test Charge in Electric Field

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \quad \text{.....................................} (2)$$

The potential difference between two points P and Q , V_{PQ}, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l} \quad \text{...............................} (3)$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as Volts.

Let us consider a point charge Q as shown in the Figure.

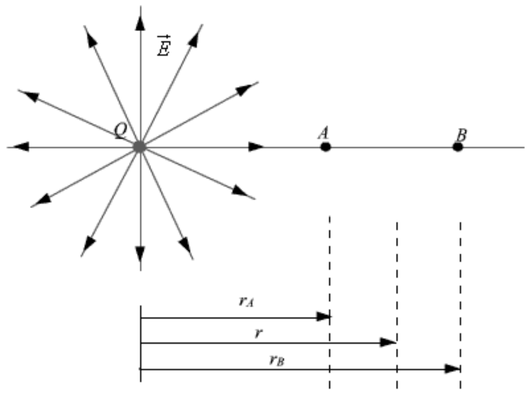


Figure: Electrostatic Potential calculation for a point charge

Further consider the two points A and B as shown in the Figure. Considering the movement of a unit positive test charge from B to A , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E} \cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B \qquad \text{...................(4)}$$

It is customary to choose the potential to be zero at infinity. Thus potential at any point (rA = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. rB = 0).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad \text{.................................. (5)}$$

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl \qquad \text{..................................(6)}$$

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. 3.

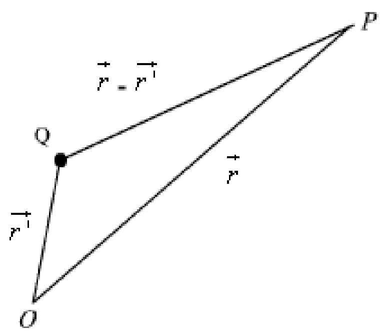


Fig. 3: Electrostatic Potential due a Displaced Charge

The potential at a point P becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} \quad \text{................................(7)}$$

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also. Let us first consider N point charges

Q1, Q2 ,..... QN located at points with position vectors

$\vec{r}_1$, $\vec{r}_2$,....... $\vec{r}_N$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{|\vec{r} - \vec{r}_1|} + \frac{Q_2}{|\vec{r} - \vec{r}_2|} + \cdots \frac{Q_N}{|\vec{r} - \vec{r}_N|}\right) \quad \text{...............................} \quad (8)$$

OR

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{|\vec{r} - \vec{r}_n|} \quad \text{...................................(9)}$$

Potential due to different Charge distributions:-

For continuous charge distribution, we replace point charges by corresponding

charge elements $\rho_l dl$ or $\rho_s ds$ or $\rho_v dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral

With these modifications we can write:

For line charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{|\vec{r}-\vec{r}_s|}$$ ....(10)

For surface charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{|\vec{r}-\vec{r}_s|}$$ ....(11)

For volume charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{|\vec{r}-\vec{r}_s|}$$ .... (12)

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$ ....(13)

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E}\cdot d\vec{l} + C$$ .... (14)

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = \frac{W}{Q}$$ ....(15)

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point P1 to P2 in one path and then from point P2 back to P1 over a different path. If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position P1. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable, the work must have to be independent of path and depends on the initial and final positions.

Potential Gradient:-

Since the potential difference is independent of the paths taken, V_{AB} = - V_{BA} , and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0 \quad \text{.................................(16)}$$

Applying Stokes's theorem, we can write:

$$\oint \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E}\right) \cdot d\vec{s} = 0 \quad \text{............................ (17)}$$

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0 \quad \text{......................(18)}$$

Any vector field that satisfies is called an irrotational field. From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial x}dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -\vec{E} \cdot d\vec{l}$$

$$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l} \quad \text{.................................(19)}$$

from which we obtain,

$$\vec{E} = -\nabla V \quad \text{.. (20)}$$

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field , i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla\times\vec{E}$.

Equipotential Surfaces

An equipotential surface refers to a surface where the potential is constant. The intersection of an equipotential surface with an plane surface results into a path called an equipotential line. No work is done in moving a charge from one point to the other along an equipotential line or surface.

In figure 4, the dashes lines show the equipotential lines for a positive point charge. By symmetry, the equipotential surfaces are spherical surfaces and the equipotential lines are circles. The solid lines show the flux lines or electric lines of force.

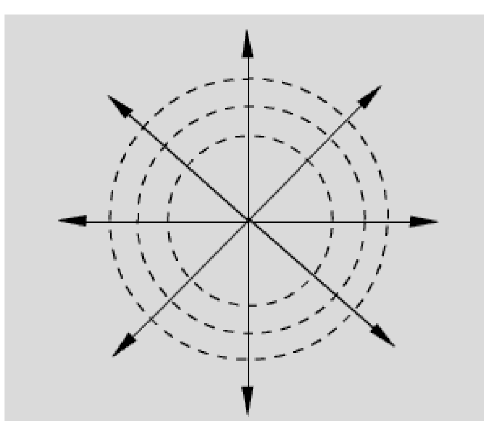


Fig.4: Equipotential Lines for a Positive Point Charge

Michael Faraday as a way of visualizing electric fields introduced flux lines. It may be seen that the electric flux lines and the equipotential lines are normal to each other. In order to plot the equipotential lines for an electric dipole, we observe that for a given Q and d, a constant V requires that $\frac{\cos\theta}{r^2}$ is a constant. From this we can write

$r = c_v\sqrt{\cos\theta}$ to be the equation for an equipotential surface and a family of surfaces can be generated for various values of cv.When plotted in 2-D this would give equipotential

lin
es.

To determine the equation for the electric field lines, we note that field lines represent the direction of $\vec{E}$ in space. Therefore,

$dl = kE$, where k is a constant ....(21)

$$\hat{a}_r dr + r d\theta \hat{a}_\theta + \hat{a}_\phi r \sin\theta = k(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = d\vec{l}$$ ....(22)

For the dipole under consideration E_ϕ =0 , and therefore we can write,

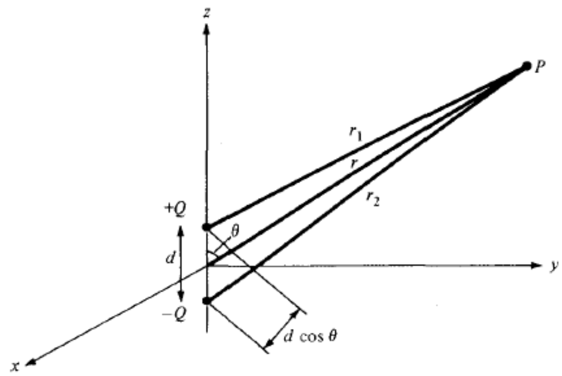
$$\frac{dr}{E_r} = \frac{r d\theta}{E_\theta}$$

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2d(\sin\theta)}{\sin\theta}$$ .... (23)

Electric Dipole

- **An electric dipole** is formed when two point charges of equal magnitude but opposite

sign are separated by a small distance.



- The potential at point *P(r,θ ,ø)* is given by

$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_o}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_o r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Electrostatic Energy and Energy Density:

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q1, Q2,......., QN are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q1 is zero. Q2 is brought in the presence of the field of Q1, the work done W1= Q_2V_{21} where V_{21} is the potential at the location of Q2 due to Q1. Proceeding in this manner, we can

write, the total work done

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \ldots\ldots\ldots\ldots\ldots + (V_{N1}Q_N + \ldots\ldots\ldots + V_{N(N-1)}Q_N) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

Had the charges been brought in the reverse order,

$$W = (V_{1N}Q_1 + \ldots\ldots\ldots + V_{12}Q_1) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (V_{(N-2)(N-1)}Q_{N-2} + V_{(N-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$$

................(25)

Therefore,

$$2W = (V_{1N} + V_{1(N-1)} + \ldots\ldots + V_{12})Q_1 + (V_{2N} + V_{2(N-1)} + \ldots\ldots + V_{23} + V_{21})Q_1 \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots + (V_{N1} + \ldots\ldots V_{N2} + V_{N(N-1)})Q_N$$

……(26)

Here V_{ij} represent voltage at the I^{th} charge location due to Jth charge. Therefore,

$$2W = V_1Q_1 + \ldots\ldots\ldots\ldots + V_NQ_N = \sum_{I=1}^{N} V_IQ_I$$ Or, $$W = \frac{1}{2}\sum_{I=1}^{N} V_IQ_I$$ ……………(27)

If instead of discrete charges, we now have a distribution of charges over a volume v

then we can write, $$W = \frac{1}{2}\int_v V\rho_v dv$$ ……………(28)

where ρ_v is the volume charge density and V represents the potential function.

Since, $\rho_v = \nabla \cdot \vec{D}$, we can write

$$W = \frac{1}{2}\int_v \left(\nabla \cdot \vec{D}\right)V dv$$ …………………………………(29)

Using the vector identity, $\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla\cdot\vec{D}$, we can write

$$W = \frac{1}{2}\int_v \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right)dv$$
$$= \frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s} - \frac{1}{2}\int_v \left(\vec{D}\cdot\nabla V\right)dv$$

---------------------------(30)

In the expression $\frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s}$, for point charges, since V varies as 1/r and D varies as $1/r^2$, the term V varies as $1/r^3$.

Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_v \left(\vec{D}.\nabla V\right)dv = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right)dv = \frac{1}{2}\int_v \left(\varepsilon E^2\right)dv = \int_v w_e dv \qquad \text{..........(31)}$$

$w_e = \frac{1}{2}\varepsilon E^2$, is called the energy density in the electrostatic field. Poisson's and Laplace's Equations

For electrostatic field, we have seen that

$$\nabla \cdot \vec{D} = \rho_v$$
$$\vec{E} = -\nabla V \qquad \text{..(32)}$$

Form the above two equations we can write

$$\nabla \cdot (\varepsilon \vec{E}) = \nabla \cdot (-\varepsilon \nabla V) = \rho_v \qquad \text{..(33)}$$

Using vector identity we can write,

$$\varepsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \varepsilon = -\rho_v \text{......(34)}$$

For a simple homogeneous medium, ε is constant and $\nabla \varepsilon = 0$. Therefore,

$$\nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \text{................(35)}$$

This equation is known as Poisson's equation. Here we have introduced a new operator

∇^2 , (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right) \qquad \text{...............(36)}$$

Therefore, in Cartesian coordinates, Poisson equation can be written as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon} \qquad \text{.............(37)}$$

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In spherical polar coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{...............(38)}$$

At points in simple media, where no free charge is present, Poisson's equation reduces to

$$\nabla^2 V = 0 \qquad \text{---------------(39)}$$

Which is known as Laplace Equation.

Laplace's and Poisson's equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found hroughout the volume. We shall consider such applications in the section where we deal with boundary value problems.

Convection and conduction current:

- The current (in amperes) through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt}$$ ----------------(40)

- We now introduce the concept of *current density* J. If current I flows through a surface *S*, the current density is

$$J_n = \frac{\Delta I}{\Delta S}$$ ---------------(41)

The current density is assumed to be perpendicular to the surface.
If the current density is not normal to the surface, then

$\Delta I = J_n \, \Delta s$-------------(42)

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$ -----------------(43)

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta \ell}{\Delta t} = \rho_v \, \Delta S \, u_y$$ ---------------------(44)

Depending on how I is produced, there are different kinds of current densities such as,

- ✓ Convection current density
- ✓ Conduction current Density
- ✓ Displacement Current Density

Convection Current Density:

- Convection current, which is different from conduction current, does not involve conductors and consequently does not satisfy Ohm's Law.
- This type of current occurs when current flows through an insulating medium such as liquid, rarefied gas or a vacuum.
- For example a beam of electrons in a vacuum tube can be considered as convection current.
- Consider a filament as shown in figure below.

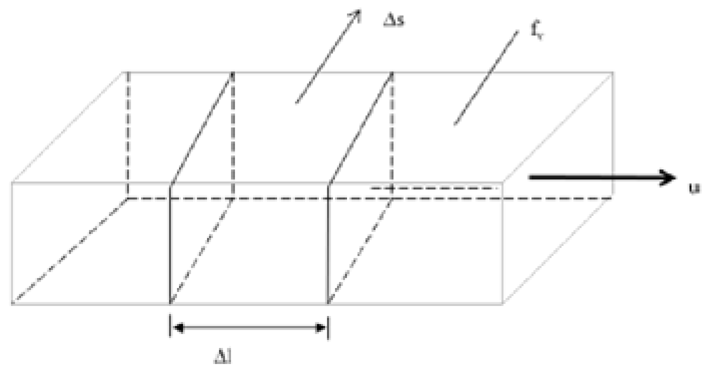


Fig: Current in a filament

- The current density m a given point is the current through a unit normal area at that point.

$$\mathbf{J} = \rho_v \mathbf{u}$$

- I is the convection current
- The conduction current is

$$\mathbf{J} = \rho_v \mathbf{u} = \frac{ne^2\tau}{m}\mathbf{E} = \sigma \mathbf{E}$$

$$J = \sigma E \qquad \text{.........................(45)}$$

UNIT-3
ELECTROSTATICS-III

Objectives:

- To introduce the conductance, boundary conditions, capacitance and Solution to Laplace's equation.

Syllabus:

UNIT – III: Electrostatics-III

Conductor properties, Polarization in Dielectrics, Boundary conditions for Dielectric – Dielectric and Conductor - Dielectric Interfaces, Capacitance - Parallel Plate, Coaxial and Spherical Capacitors, Poisson's and Laplace's equations, Examples of the solution of Laplace's equation (Direct Integration Method for One dimensional Potential Variation Problems).

Outcomes:

Students will be able to

- Understand polarization.
- Derive the boundary conditions at different interfaces..
- Understand parallel plate, coaxial and spherical capacitance working.
- Solve the laplace's equation.

Continuity Equation and Relaxation Time

Let us consider a volume V bounded by a surface S. A net charge Q exists within this region. If a net current I flows across the surface out of this region, from the principle of conservation of charge this current can be equated to the time rate of decrease of charge within this volume. Similarly, if a net current flows into the region, the charge in the volume must increase at a rate equal to the current. Thus we can write,

$$I = -\frac{dQ}{dt}$$

or, $$\oint_S \vec{J}.d\vec{s} = -\frac{d}{dt}\int_V \rho dv$$

Applying divergence theorem we can write,

$$\int_V \nabla.\vec{J} dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

It may be noted that, since ρ in general may be a function of space and time, partial derivatives are used. Further, the equation holds regardless of the choice of volume V , the integrands must

be equal.

Therefore we can write,

$$\nabla.\vec{J} = -\frac{\partial \rho}{\partial t}$$

The equation (48) is called the continuity equation, which relates the divergence of current density vector to the rate of change of charge density at a point.

For steady current flowing in a region, we have

$$\nabla.\vec{J} = 0$$

Considering a region bounded by a closed surface,

$$\oint_S \vec{J}.d\vec{s} = 0$$

which can be written as,

$$\sum_i I_i = 0$$

when we consider the close surface essentially encloses a junction of an electrical circuit.

which is called the *continuity of current equation*

$$T_r = \frac{\varepsilon}{\sigma}$$

- The time constant *Tr* (in seconds) is known as the *relaxation time* or *rearrangement time.*
- **Relaxation time** is the lime it takes u charge placed in the interior of a material to drop to e^{-1}= 36.8 percent of its initial value.

Conductor Properties:

Materials are broadly classified in terms of their electrical properties as conductors and nonconductors. Non conducting materials are usually referred to as insulators or dielectrics.

Conductor Properties:

In a broad sense, materials may be classified in terms of their conductivity **σ**, in mhos per meter (℧ /m) or Siemens per meter (S/m), as conductors and nonconductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency. A material with high conductivity (**σ** » 1) is referred to as a metal whereas one with low conductivity (**σ** << 1) is referred to as an insulator. A material whose conductivity lies somewhere between those of metals and insulators is called a semiconductor.

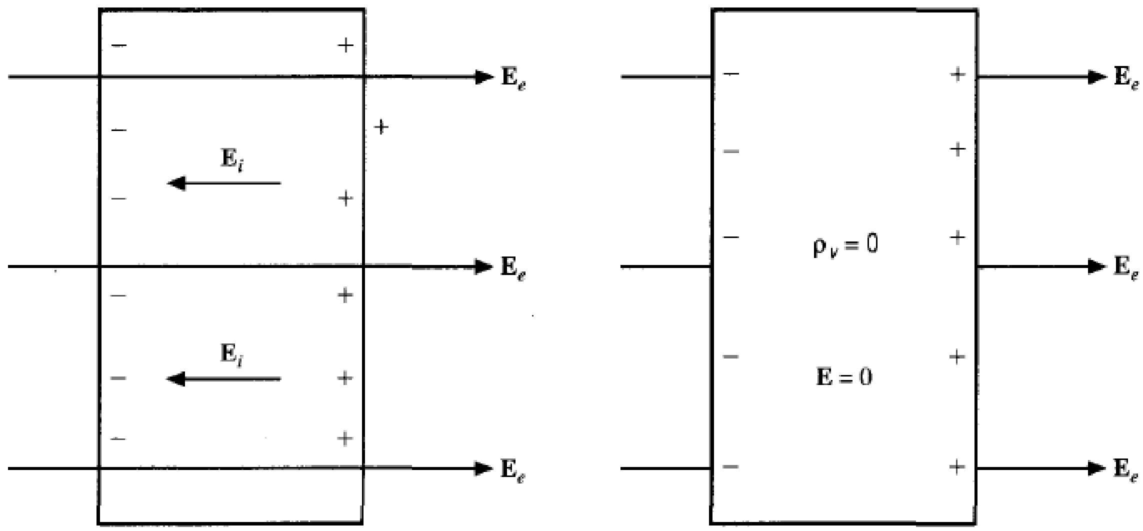
A conductor has abundance of charge that is free to move. When an external electric field E_e is applied, the positive free charges are pushed along the same direction as the applied field, while the negative free charges move in the opposite direction. This charge migration takes place very quickly. The free charges do two things. First, they accumulate on the surface of the conductor and form an induced surface charge. Second, the induced charges set up an internal induced field E,, which cancels the externally applied field E_e.

Note: A perfect conductor cannot contain an electrostatic field within it.

A conductor is called an equipotential body, implying that the potential is the same everywhere in the conductor. This is based on the fact that **E** = - ∇V = 0.

Another way of looking at this is to consider Ohm's law, **J** = σ **E**. To maintain a finite current density **J**, in a perfect conductor (σ —> ∞), requires that the electric field inside the conductor must vanish. In other words, **E** —> 0 because $\sigma \longrightarrow \infty$ in a perfect conductor. If some charges are introduced in the interior of such a conductor, the charges will move to the surface and redistribute themselves quickly in such a manner that the field inside the conductor vanishes. According to Gauss's law, if **E** = 0, the charge density ρ_v must be zero.

$$\mathbf{E} = 0, \qquad \rho_v = 0, \qquad V_{ab} = 0 \qquad \text{inside a conductor}$$



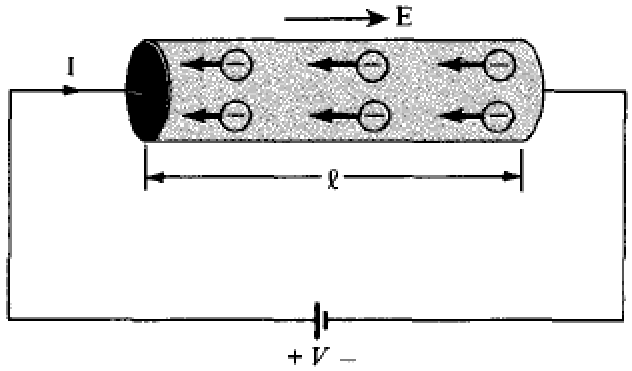
(a) (b)

An isolated conductor under the influence of an applied field; **(b)** a conductor has zero electric field under static conditions.

The conductor has a *uniform* cross section *S* and is of length l. The direction of the electric field E produced is the same as the direction of the flow of positive charges or current I. This direction is opposite to the direction of the flow of electrons. The electric field applied is uniform and its magnitude is given by E=V/ l

Since the conductor has uniform cross section J = I/S.

And R=V/I . Therefore R= (ρ_c l) / S



A conductor of uniform cross section under an applied **E** field.

On solving

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\int \sigma\mathbf{E} \cdot d\mathbf{S}}$$

And ,

$$P = \int \mathbf{E} \cdot \mathbf{J}\, dv$$

Which is known as Joules law.

$$w_P = \frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} = \sigma \left|\mathbf{E}\right|^2$$

$$P = \int_L E\, dl \int_S J\, dS = VI$$

$$P = I^2R$$

Which is the common form of Joule's law in electromagnetic theory.

POLARIZATION IN DIELECTRICS:

When an electric field **E** is applied, the positive charge is displaced from its equilibrium position in the direction of **E** by the force $\mathbf{F}_+ = Q\mathbf{E}$ while the negative charge is displaced in the opposite direction by the force $\mathbf{F}_- = Q\mathbf{E}$. *A* dipole results from the displacement of the charges and the dielectric is said to be *polarized.* In the polarized state, the electron cloud is distorted by the applied electric field E. This distorted charge distribution is equivalent, by the principle of superposition, to the original distribution plus a dipole whose moment is **p** = $Q\boldsymbol{d}$ dipole moment per unit volume of the dielectric is

$$\mathbf{P} = \frac{\lim_{\Delta v \to 0} \sum_{k=1}^{N} Q_k \mathbf{d}_k}{\Delta v}$$

The electric field **E** on a dielectric is the creation of dipole moments that align themselves in the direction of **E**. This type of dielectric is said to be non-polar.

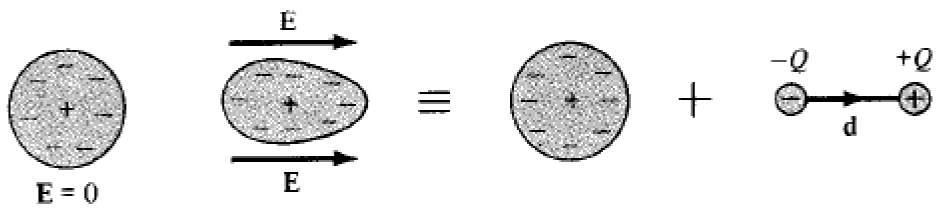
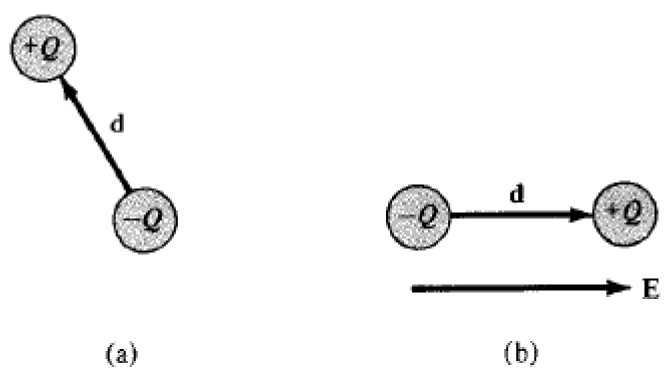


Fig.: Polarization of a nonpolar atom or molecule.



Polarization of a polar molecule:
(a) permanent dipole (**E** = 0), **(b)** alignment of permanent dipole (**E** ≠ 0).

BOUNDARY CONDITIONS:

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called *boundary conditions.*

These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. Obviously, the conditions will be dictated by the types of material the media are made of. We shall consider the boundary conditions at an interface separating

• dielectric (Єr1) and dielectric (Єr2)

• conductor and dielectric

• conductor and free space

To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

Also we need to decompose the electric field intensity E into two orthogonal components:
$\mathbf{E}=\mathbf{E}_t+\mathbf{E}_n$
where **E**, and ***En*** are, respectively, the tangential and normal components of E to the interface of interest.
Dielectric-Dielectric Boundary Conditions:
Consider the E field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$. ***E1*** and **E2** in media 1 and 2, respectively, can be decomposed as

$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n}$

$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n}$

We apply to the closed path *abcda* assuming that the path is very small with respect to the variation of **E**.

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

$E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E}_n|$. As $\Delta h \rightarrow 0$,

$$\boxed{E_{1t} = E_{2t}}$$

Thus the tangential components of E are the same on the two sides of the boundary In other words, E, undergoes no change on the boundary and it is said to be *continuous* across the boundary. Since D = εE = Dt + Dn, equation can be written as

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

that is, ***D***, undergoes some change across the interface. Hence ***D,*** is said to be *discontinuous* across the interface.

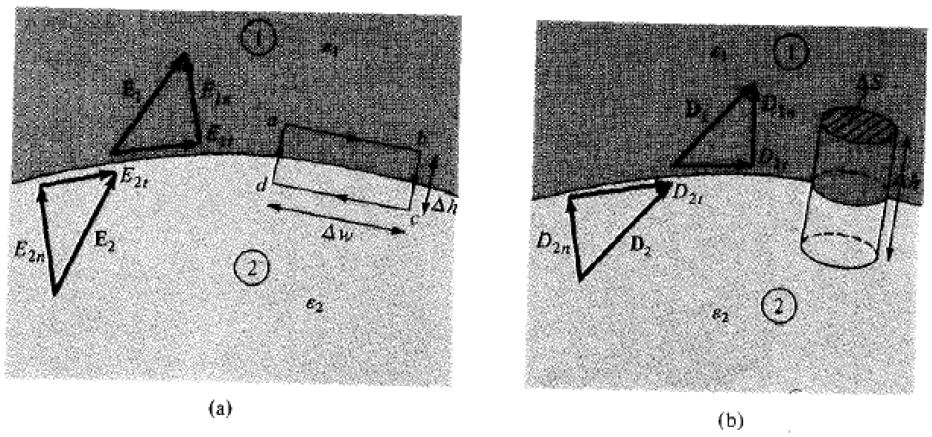


Fig.: Dielectric-Dielectric Boundary

Similarly, we can apply to the pillowbox (Gaussian Surface). Allowing Δh —> 0

$$\Delta Q = \rho_S \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

$$\boxed{D_{1n} - D_{2n} = \rho_S}$$

where p_s is the free charge density placed deliberately at the boundary. The assumption is that **D** is directed from region 2 to region 1 and must be applied accordingly. If no free charges exist at the interface (i.e., charges are not deliberately placed there), p_s = 0.Then $\mathbf{D_{1n}=D_{2n}}$.

Thus the normal component of **D** is continuous across the interface; that is, D_n undergoes no change at the boundary. Since D = εE. Therefore, $\varepsilon_1 E_{1n}=\varepsilon_2 E_n$. It shows that the normal components of **E** is discontinuous at the boundary.

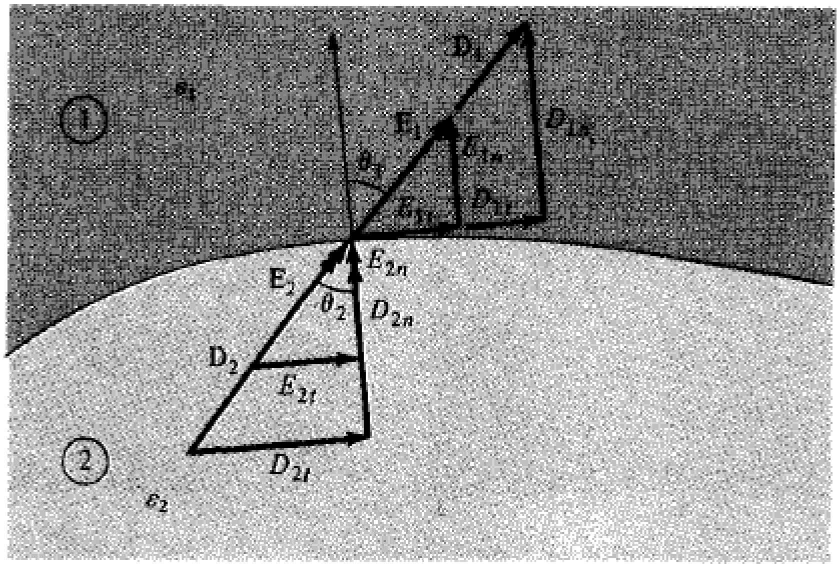


Fig.: Refraction of **D** and **E** at a Dielectric- dielectric boundary.

On solving,

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

This is the law of refraction of the electric field at a boundary free charge.

Conductor-Dielectric Boundary Conditions:

The conductor is assumed to be perfect (i.e., $\sigma \longrightarrow \infty$ or $\rho_c \longrightarrow 0$). Although such a conductor is not practically realizable, we may regard conductors such as copper and silver as though they were perfect conductors.

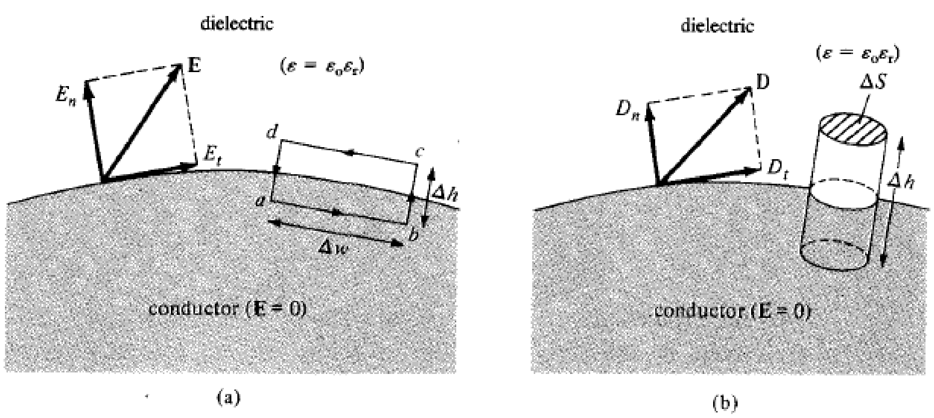


Fig.: Conductor-Dielectric Boundary

To determine the boundary conditions for a conductor-dielectric interface, we follow the same procedure used for dielectric-dielectric interface except that we incorporate the fact that **E** = 0 inside the conductor. For the closed path *abcda* of Figure above gives

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$, $E_t = 0$

Similarly, for the pillbox by letting $\Delta h \to 0$, we get

$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S$$

On solving $D_n = \rho_s$.

Conductor-Free Space Boundary Conditions:

This is a special case of the conductor-dielectric conditions and is illustrated. The boundary conditions at the interface between a conductor and free space can be obtained BY replacing ε_r by 1 (because free space may be regarded as a special dielectric for which $\varepsilon_r = 1$). We expect the electric field **E** to be external to the conductor and normal to its surface. Thus the boundary conditions are

$$D_t = \varepsilon_o E_t = 0, \qquad D_n = \varepsilon_o E_n = \rho_S$$

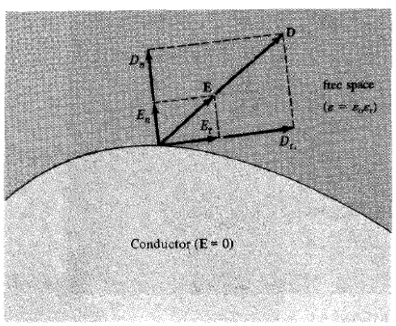


Fig.: Conductor – free space boundary

CAPACITANCE AND CAPACITORS

We have already stated that a conductor in an electrostatic field is an Equipotential body and any charge given to such conductor will distribute themselves in such a manner that electric field inside the conductor vanishes. If an additional amount of charge is supplied to an isolated conductor at a given potential, this additional charge will increase the surface charge density ρ_s

. Since the potential of the conductor is given by $V = \frac{1}{4\pi\varepsilon_0} \int_{s'} \frac{\rho_s ds'}{r}$, the potential of conductor will also increase maintaining the ratio same $\frac{Q}{V}$.

Thus we can write $C = \frac{Q}{V}$ where the constant of proportionality C is called the capacitance of the isolated conductor. SI unit of

Capacitance is Coulomb/ Volt also called Farad denoted by F. It can be seen that if V=1, C= Q. Thus capacity of an isolated conductor can also be defined as the amount of charge in Coulomb required to raise the potential of the conductor by 1 Volt.

Of considerable interest in practice is a capacitor that consists of two (or more) conductors carrying equal and opposite charges and separated by some dielectric media or free space. The conductors may have arbitrary shapes. A two-conductor capacitor is shown in figure below.

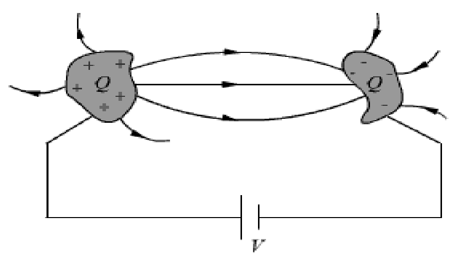


Fig. : Capacitance and Capacitors

When a d-c voltage source is connected between the conductors, a charge transfer occurs which results into a positive charge on one conductor and negative charge on the other conductor. The conductors are equipotential surfaces and the field lines are perpendicular to the conductor surface. If V is the mean potential difference between the conductors, the capacitance is given by

$$C = \frac{Q}{V}$$

Capacitance of a capacitor depends on the geometry of the conductor and the permittivity of the medium between them and does not depend on the charge or potential difference between conductors. The capacitance can be computed by assuming Q(at the same time -Q on the other conductor), first determining $\vec{E}$ using Gauss's theorem and then determining $V = -\int \vec{E}.d\vec{l}$.

We illustrate this procedure by taking the example of a parallel plate capacitor.

Series and parallel Connection of capacitors

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 1. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2}$$

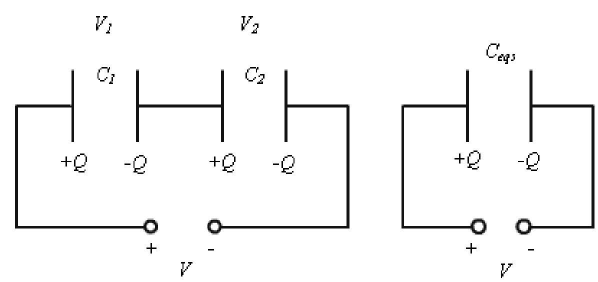


Fig.: Series Connection of Capacitors

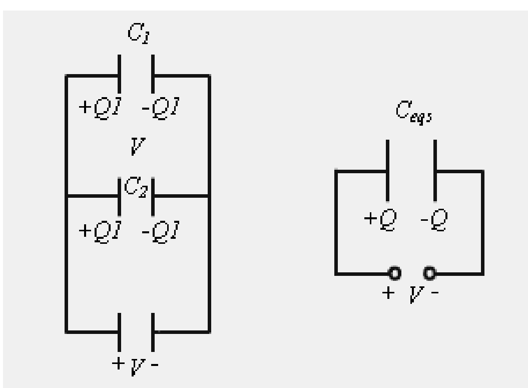


Fig. 2: Parallel Connection of Capacitors

The same approach may be extended to more than two capacitors connected in series. Parallel Case: For the parallel case, the voltages across the capacitors are the same. The total charge

$$C_{eqp} = \frac{Q}{V} = C_1 + C_2$$

$$Q = Q_1 + Q_2 = C_1V + C_2V$$

Therefore,

B. Coaxial Capacitor

This is essentially a coaxial cable or coaxial cylindrical capacitor. Consider length *L* of two coaxial conductors of inner radius *a* and outer radius *b (b > a)* as shown in Figure. Let the space between the conductors be filled with a homogeneous dielectric with permittivitys. We assume that conductors 1 and 2, respectively, carry +*Q* and -*Q* uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius *p (a < p < b),* we obtain

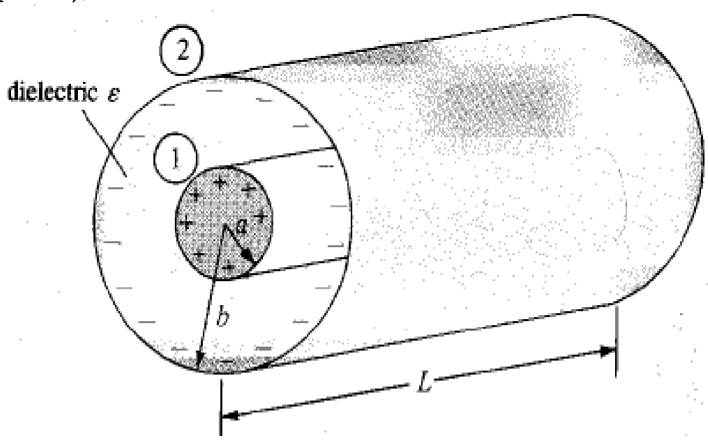


Figure : Coaxial capacitor.

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_\rho 2\pi\rho L$$

Hence

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon\rho L}\mathbf{a}_\rho$$

Thus the capacitance of a coaxial cylinder is given by

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

C. Spherical Capacitor:

This is the case of two concentric spherical conductors. Consider the inner sphere of radius *a* and outer sphere of radius *b{b> a)* separated by a dielectric medium with permittivity as shown in Figure. We assume charges +*Q* and -*Q* on the inner and outer spheres

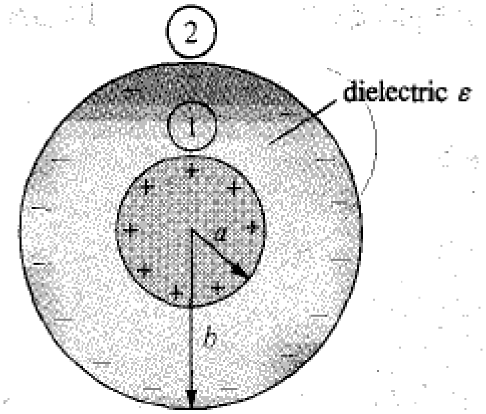


Figure: Spherical capacitor

respectively. By applying Gauss's law to an arbitrary Gaussian spherical surface of radius

$r(a<r<b)$,

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_r 4\pi r^2$$

that is

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\mathbf{a}_r$$

Thus the capacitance of the spherical capacitor is

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$$

POISSON'S AND LAPLACE'S EQUATIONS:

Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear material medium)

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \rho_v$$

And

$$\mathbf{E} = -\nabla V$$

By above equations

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

for an inhomogeneous medium. For a homogeneous medium, above equation becomes

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

This is known as *Poisson's equation. A* special case of this equation occurs when pv = 0
(i.e., for a charge-free region).

$\nabla^2 V = 0$ Which is known as *Laplace's equation.*

Recall that the Laplacian operator ∇^2 was derived.Thus Laplace's equation in Cartesian, cylindrical,or spherical coordinates respectively is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

GENERAL PROCEDURE FOR SOLVING POISSON'S OR LAPLACE'S EQUATION:

The following general procedure may be taken in solving a given boundary-value problem
involving Poisson's or Laplace's equation:

1. Solve Laplace's (if pv = 0) or Poisson's equation using either

(a) direct integration when *V* is a function of one variable, or

(b) separation of variables if *V* is a function of more than one variable. The solution at this point is not unique but expressed in terms of unknown integration constants to be determined.

2. Apply the boundary conditions to determine a unique solution for *V.* Imposing the
given boundary conditions makes the solution unique.

3. Having obtained *V,* find E using E = -∆V and D from D = ƐE.

4. If desired, find the charge Q induced on a conductor using Q = J p_s dS where ps — Dn and Dn is the component of D normal to the conductor. If necessary, the capacitance between two conductors can be found using $C = Q/V$.

Solving Laplace's (or Poisson's) equation, as in step 1, is not always as complicated as it may seem. In some cases, the solution may be obtained by mere inspection of the problem. Also a solution may be checked by going backward and finding out if it satisfies both Laplace's (or Poisson's) equation and the prescribed boundary conditions.

EXAMPLE

Current-carrying components in high-voltage power equipment must be cooled to carry away the heat caused by ohmic losses. A means of pumping is based on the force transmitted to the cooling fluid by charges in an electric field. The electro hydrodynamic (EHD) pumping is modeled in Figure, The region between the electrodes contains a uniform charge p0, which is generated at the left electrode and collected at the right electrode. Calculate the pressure of the pump if po = 25 mC/m3 and Vo = 22 kV.

Solution:

We apply Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

The boundary conditions $V(z = 0) = Vo$ and $V(z = d) = 0$ show that V depends only on z
(there is no p or <j> dependence). Hence

$$\frac{d^2V}{dz^2} = \frac{-\rho_0}{\varepsilon}$$

Integrating once gives

$$\frac{dV}{dz} = \frac{-\rho_0 z}{\varepsilon} + A$$

Integrating again yields

$$V = -\frac{\rho_o z^2}{2\varepsilon} + Az + B$$

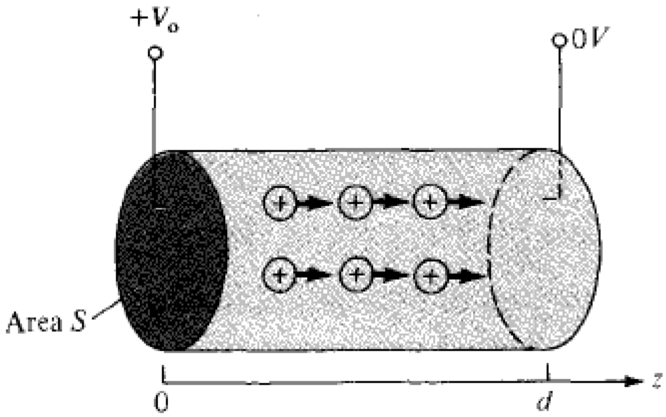


Figure: An electrohydrodynamic pump where *A* and *B* are integration constants to be determined by applying the boundary conditions.

When $z = 0$, $V = Vo$,

$$V_o = -0 + 0 + B \rightarrow B = V_o$$

When $z = d$, $V = 0$,

$$0 = -\frac{\rho_o d^2}{2\varepsilon} + Ad + V_o$$

or

$$A = \frac{\rho_o d}{2\varepsilon} - \frac{V_o}{d}$$

The electric field is given by

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = \left(\frac{\rho_o z}{\varepsilon} - A\right)\mathbf{a}_z$$

$$= \left[\frac{V_o}{d} + \frac{\rho_o}{\varepsilon}\left(z - \frac{d}{2}\right)\right]\mathbf{a}_z$$

The net force is

$$\mathbf{F} = \int \rho_v \mathbf{E}\, dv = \rho_o \int dS \int_{z=0}^{d} \mathbf{E}\, dz$$

$$= \rho_o S \left[\frac{V_o z}{d} + \frac{\rho_o}{2\varepsilon}(z^2 - dz)\right]\Bigg|_0^d \mathbf{a}_z$$

$$\mathbf{F} = \rho_o S V_o \mathbf{a}_z$$

The force per unit area or pressure is

$$\rho = \frac{F}{S} = \rho_o V_o = 25 \times 10^{-3} \times 22 \times 10^3 = 550 \text{ N/m}^2$$

Unit – 4

Objectives:

- To introduce the concepts of magnetic fields, potential and force on different elements in magnetic field.

Syllabus:

UNIT – IV: Magnetostatics-I

Biot-Savart's Law, Ampere's Circuital Law-Applications of Ampere's Circuital Law : Infinite Line Current, Infinite Sheet of Current, Magnetic Flux and Magnetic Flux Density, Magnetic Scalar and Vector Potentials, Force on a moving charge- Lorentz Force Equation, Force on a current element.

Outcomes:

Students will be able

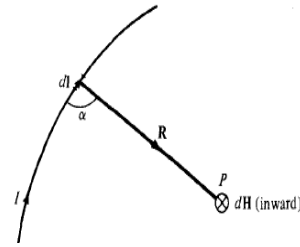
- Understand the Biot-Savart law, Ampere's law and stokes theorem.
- Calculate the magnetic field intensity at different currents.
- Understand the magnetic field and magnetic flux density.
- Measure the force on different elements.

Biot-Savart's Law

The magnetic field intensity *dH* produced at a point *P*, by the differential current clement I *dl* is proportional to the product I *dl* and the sine of the angle *α* between the clement and the line joining *P* to the element and is inversely proportional to the square of the distance *R* between *P* and the element.

$$dH \propto \frac{I\,dl \sin\alpha}{R^2}$$

$$dH = \frac{kI\,dl \sin\alpha}{R^2}$$

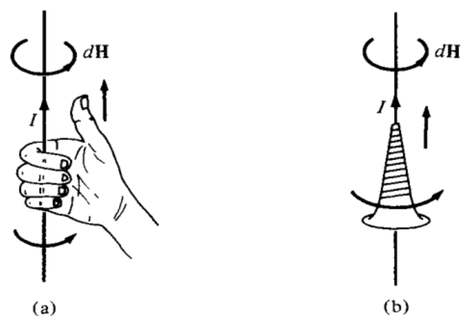


where k is the constant of proportionality. In SI units, $k = 1/4\pi$,

From the cross product

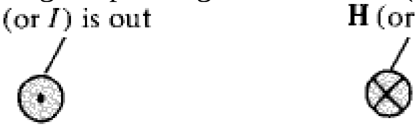
$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

We can use the right-handed screw rule to determine the direction of *dH:* with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of *dH* as in Figure



Determining the direction of *dH* using
(a) the right-hand rule, or (b) the right-handed screw rule.

It is customary to represent the direction of the magnetic field intensity H (or current /) by a small circle with a dot or cross sign depending on whether H (or *I)* is out of, or into.



If we define K as the surface current density (in amperes/meter) and J as the volume current density (in amperes/meter square), the source elements are related as

$$I\,d\mathbf{l} \equiv \mathbf{K}\,dS \equiv \mathbf{J}\,dv$$

Thus in terms of the distributed current sources, the Biot-Savart law as becomes

$$\mathbf{H} = \int_L \frac{I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(line current)}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\, dS \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(surface current)}$$

$$\mathbf{H} = \int_v \frac{\mathbf{J}\, dv \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(volume current)}$$

Ampere's circuit law states that the line integral of the tangential component of H around a *dosed* path is the same as the net current I_{enc} enclosed by the path.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}$$

Ampere's law is similar to Gauss's law and it is easily applied to determine H when the current distribution is symmetrical.
By applying Stoke's theorem to the left-hand side

$$I_{\text{enc}} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Applications of Ampere's Law

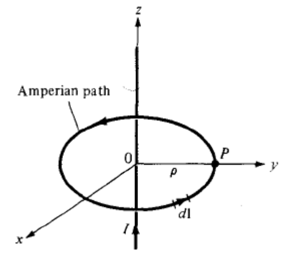
We now apply Ampere's circuit law to determine H for some symmetrical current distributions as we did for Gauss's law.

Infinite Line Current

Consider an infinitely long filamentary current I along the z-axis. To determine H at an observation point *P*, we allow a closed path pass through *P*. This path, on which Ampere's law is to be applied, is known as an *Amperian path* (analogous to the term Gaussian surface).

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho\, d\phi\, \mathbf{a}_\phi = H_\phi \int \rho\, d\phi = H_\phi \cdot 2\pi\rho$$

$$\mathbf{H} = \frac{I}{2\pi\rho}\, \mathbf{a}_\phi$$

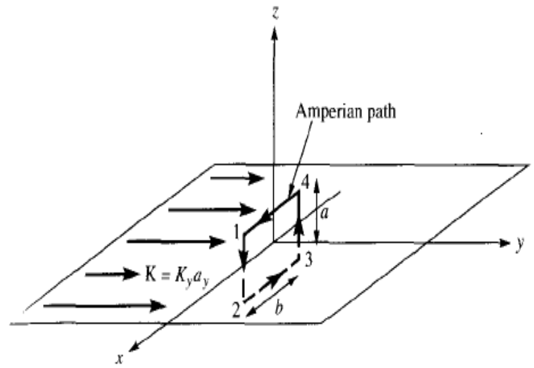


Infinite Sheet of Current

Consider an infinite current sheet in the $z = 0$ plane. If the sheet has a uniform current density K = *Kyay* A/m

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = K_y b$$

$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases}$$



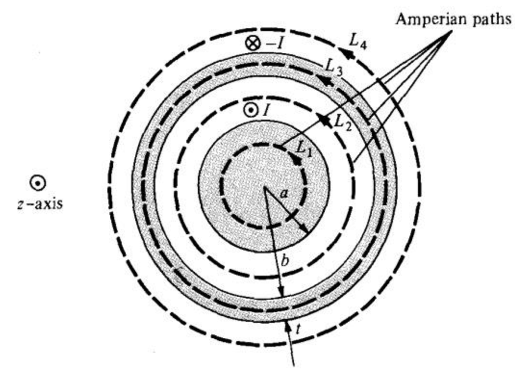
$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$$

$$\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n$$

Where $\mathbf{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

Infinitely Long Coaxial Transmission Line

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The inner conductor has radius *a* and carries current I while the outer conductor has inner radius *b* and thickness *t* and carries return current -I . We want to determine H everywhere assuming that current is uniformly distributed in both conductors.



$$\mathbf{H} = \begin{cases} \frac{I\rho}{2\pi a^2} \mathbf{a}_\phi, & 0 \le \rho \le a \\ \frac{I}{2\pi\rho} \mathbf{a}_\phi, & a \le \rho \le b \\ \frac{I}{2\pi\rho} \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] \mathbf{a}_\phi, & b \le \rho \le b + t \\ 0, & \rho \ge b + t \end{cases}$$

The **curl** of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$

Stokes's theorem suites that the circulation of a vector field A around a (closed) pain L is equal to the surface integral of the curl of A over the open surface *S* bounded by L provided that A and $\nabla \times \mathbf{A}$ are continuous on .V

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Magnetic Flux Density

The magnetic flux density B is similar to the electric flux density D. As D = *μoE* in free space, the magnetic flux density B is related to the magnetic field intensity H according to

$$\mathbf{B} = \mu_o \mathbf{H}$$

Where μo is a constant known as the *permeability of free space.* The constant is in henrys/meter (H/m) and has the value of

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m}$$

The magnetic flux through a surface *S* is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

An **isolated magnetic** charge does not exist. Thus the total flux through a closed surface in a magnetic field must be zero.

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

By divergence theorem

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \, dv = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

This equation is the fourth Maxwell's equation to be derived

TABLE 7.2 Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservativeness of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's law

Magnetic Scalar and Vector Potentials

Recall that some electrostatic field problems were simplified by relating the electric potential *V* to the electric field intensity E (E = — VV). Similarly, we can define a potential associated with magnetostatic field B. In fact, the magnetic potential could be scalar *Vm* or vector A. To define *Vm* and A involves recalling two important identities

$$\nabla \times (\nabla V) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

We define the *magnetic scalar potential Vm* (in amperes) as related to H according to

$$\boxed{\mathbf{H} = -\nabla V_m} \qquad \text{if } \mathbf{J} = 0$$

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = 0$$

We can define the *vector magnetic potential* A (in Wb/m) such that

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$V = \int \frac{dQ}{4\pi\varepsilon_o r}$$

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int_L \frac{I\, d\mathbf{l}' \times \mathbf{R}}{R^3}$$

Forces due to Magnetic Fields

The force can be (a) due to a moving charged particle in a B field, (b) on a current element in an external B field, or (c) between two current elements.

Force on a Charged Particle

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force *Fm* experienced by a charge *Q* moving with a velocity u in a magnetic field B is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

This clearly shows that Fm is perpendicular to both u and B.

For a moving charge *Q* in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

This is known as the *Lorentz force equation.* It relates mechanical force to electrical force. If the mass of the charged particle moving in E and B fields is *m,* by Newton's second law of motion.

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Force on a Current Element

To determine the force on a current element / *dl* of a current-carrying conductor due to the magnetic field B, ``

$$\mathbf{J} = \rho_v \mathbf{u}$$

$$I\, d\mathbf{l} = \mathbf{K}\, dS = \mathbf{J}\, dv$$

$$I\, d\mathbf{l} = dQ\, \mathbf{u}$$

This shows that an elemental charge *dQ* moving with velocity u (thereby producing convection current element *dQ* u) is equivalent to a conduction current element I*dl.* Thus the force on a current element I*dl* in a magnetic field B

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$$

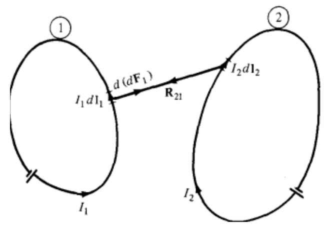
If the current / is through a closed path *L* or circuit, the force on the circuit is given by

$$\mathbf{F} = \oint_L I\, d\mathbf{l} \times \mathbf{B}$$

$$\mathbf{F} = \int_S \mathbf{K}\, dS \times \mathbf{B} \qquad \text{or} \qquad \mathbf{F} = \int_v \mathbf{J}\, dv \times \mathbf{B}$$

The **magnetic field B** is defined as the force per unit current element.

Force between Two Current Elements



Let us now consider the force between two elements $I_1\, dl_1$ and $I_2\, dl_2$ According to Biot-Savart's law, both current elements produce magnetic fields. So we may find the force *d(df)* on element I1 *dl1* due to the field *dB2* produced by element $I2\, dl_2$.

$$d(d\mathbf{F}_1) = I_1\, d\mathbf{l}_1 \times d\mathbf{B}_2$$

$$d\mathbf{B}_2 = \frac{\mu_o I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2}$$

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2}$$

We obtain the total force F, on current loop 1 due to current loop 2

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2}$$

Unit – 5

Objectives:

- To introduce the concepts of time varying fields and Maxwell's equations

Syllabus:

UNIT – V: Magnetostatics-II
Magnetic Dipole and Dipole Moment, Magnetic boundary conditions, Magnetic Energy.

Time Varying Fields and Maxwell's Equations: Faraday's law, Transformer EMF and Motional EMF, Inconsistency of Ampere's Law, Displacement current, Maxwell's equations, Time Harmonic Fields, Maxwell's Equations using Phasor Notation.

Electromagnetic waves-I: Wave Equations for Perfect Dielectrics and Conducting medium, Uniform plane wave propagation, Uniform Plane waves, Relation between E and H in a uniform Plane Wave.

Outcomes:

Students will be able

- Derive the magnetic boundary conditions at different interfaces
- Understand concepts of time varying fields
- Understand the physical significance of the Maxwell's equations
- Learn how the propagation of EM waves in different media.

TIME VARYING AND MAXWELL'S EQUATIONS:

Faraday's Law:

After Oersted's experimental discovery (upon which Biot-Savart and Ampere based their laws) that a steady current produces a magnetic field, it seemed logical to find out if magnetism would produce electricity. In 1831, about 11 years after Oersted's discovery, Michael Faraday in London and Joseph Henry in New York discovered that a time-varying magnetic field would produce an electric current.

According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying field produces an induced voltage (called *electromotive force* or simply emf) in a closed circuit, which causes a flow of current.

Faraday discovered that the **induced emf.** *M* (in volts), in any closed **circuit** is equal to the time rale of change of the magnetic flux linkage by the circuit.

This is called *Faraday's law,* and it can be expressed as

$$V_{\text{emf}} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

Where *N* is the number of turns in the circuit and V is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it.

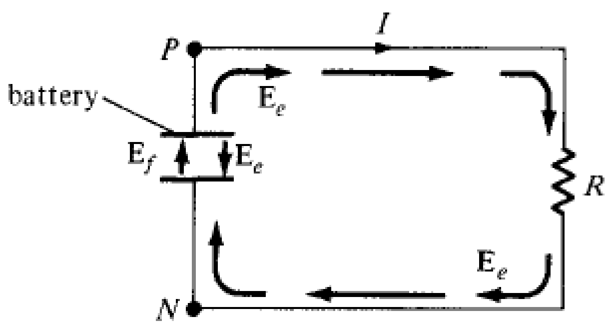


Figure: A circuit showing emf-producing field and electrostatic field E.

This is known as *Lenz's law,2* and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the original magnetic field. Recall that we described an electric field as one in which electric charges experience force. The electric fields considered so far are caused by electric charges; in such fields, the flux lines begin and end on the charges. However, there are other kinds of electric fields not directly caused by electric charges. These are emf-produced fields. Sources of emf include electric generators, batteries, thermocouples, fuel cells, and photovoltaic cells, which all convert nonelectrical energy into electrical energy.

Consider the electric circuit of Figure, where the battery is a source of emf. The electrochemical action of the battery results in an emf-produced field Ey. Due to the accumulation of charge at the battery terminals, an electrostatic field *Ee{* = — VV) also exists.

The total electric field at any point is

$E = E_y + E_e$

Note that Ey is zero outside the battery, Ey and *Ee* have opposite directions in the battery, and the direction of Ee inside the battery is opposite to that outside it. If we integrate eq. over the closed circuit,

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \oint_L \mathbf{E}_f \cdot d\mathbf{l} + 0 = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} \qquad \text{(through battery)}$$

where *§ Ee* • *d* = 0 because Ee is conservative. The emf of the battery is the line integral of the emf-produced field; that is,

$$V_{\text{emf}} = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} = -\int_N^P \mathbf{E}_e \cdot d\mathbf{l} = IR$$

since Ey and *Ee* are equal but opposite within the battery. It may also be regarded as the potential difference *(VP - VN)* between the battery's open-circuit terminals. It is important to note that:

1. An electrostatic field *Ee* cannot maintain a steady current in a closed circuit since *$LEe-dl* = 0
2. An emf-produced field Eyis nonconservative.
3. Except in electrostatics, voltage and potential difference are usually not equivalent.

TRANSFORMER AND MOTIONAL EMFs

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn *(N* = 1),

$$V_{\text{emf}} = -\frac{d\Psi}{dt}$$

In terms of E and B, eq. can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

where P has been replaced by *Js* B • *dS* and *S* is the surface area of the circuit bounded by the closed path *L*. It is clear from eq. (9.5) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that *d* and JS in eq. (9.5) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in above Figure . The variation of flux with time as in above equ may be caused in three ways:

1. By having a stationary loop in a time-varying B field
2. By having a time-varying loop area in a static B field
3. By having a time-varying loop area in a time-varying B field.

Each of these will be considered separately.

Stationary Loop in Time-Varying B transformer emf.

This is the case portrayed in Figure above where a stationary conducting loop is in a timevarying magnetic B field. Equation becomes

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

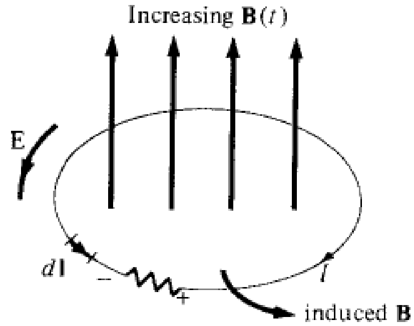


Figure.Induced emf due to a stationary loop in a timevarying B field

This emf induced by the time-varying current (producing the time-varying B field) in a stationary loop is often referred to as *transformer emf* in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in below eq. we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is one of the Maxwell's equations for time-varying fields. It shows that the timevarying E field is not conservative (V X E = 0). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time varying electric field, for example, is due to the energy from the time-varying magnetic field. Observe that Figure below, obeys Lenz's law; the induced current / flows such as to produce a magnetic field that opposes B(f).

Moving Loop in Static B Field (Motional emf)

When a conducting loop is moving in a static B field, an emf is induced in the loop. We recall from below eq, that the force on a charge moving with uniform velocity u in a magnetic field B is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

We define the *motional electric field* Em as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

If we consider a conducting loop, moving with uniform velocity u as consisting of a large number of free electrons, the emf induced in the loop is

$$V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

This type of emf is called *motional emf* or *flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators. Figure below illustrates a two-pole dc machine with one armature coil and a twobar commutator. Although the analysis of the d.c. machine is beyond the scope of this text, we can see that voltage is generated as the coil rotates within the magnetic field. Another example of motional emf is illustrated in Figure below, where a rod is moving between a pair of rails. In this example, B and u are perpendicular.
eq. becomes

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

$$\boxed{\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})}$$

Inconsistency of Ampere's Law and Displacement current:

In the previous section, we have essentially reconsidered Maxwell's curl equation for electrostatic fields and modified it for time-varying situations to satisfy Faraday's law. We shall now reconsider Maxwell's curl equation for magnetic fields (Ampere's circuit law) for Time-varying conditions.

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J}$$

But the divergence of the curl of any vector field is identically zero .
Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}$$

The continuity of current in eq, however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

where *id* is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d$$

In order for eq.

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}$$

By above equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

MAXWELL'S EQUATIONS IN FINAL FORMS:

James Clerk Maxwell (1831-1879) is regarded as the founder of electromagnetic theory in its present form. Maxwell's celebrated work led to the discovery of electromagnetic waves. Through his theoretical efforts over about 5 years (when he was between 35 and 40), Maxwell published the first unified theory of electricity and magnetism. The theory comprised all previously known results, both experimental and theoretical, on electricity and magnetism. It further introduced displacement current and predicted the existence of electromagnetic waves. Maxwell's equations were not fully accepted by many scientists until they were later confirmed by Heinrich Rudolf Hertz (1857-1894), a German physics professor. Hertz was successful in generating and detecting radio waves. The laws of electromagnetism that Maxwell put together in the form of four equations were presented in Table below for static conditions. The more generalized forms of these equations are those for time-varying conditions shown in Table 9.1. We notice from the table that the divergence equations remain the same while the curl equations have been modified. The integral form of Maxwell's equations depicts the underlying physical laws, whereas the differential form is used more frequently in solving problems. For a field to be "qualified" as an electromagnetic field, it must satisfy all four Maxwell's equations. The importance of Maxwell's equations cannot be overemphasized because they summarize all known laws of electromagnetism. We shall often refer to them in the remaining part of this text.

Since this section is meant to be a compendium of our discussion in this text, it is worthwhile to mention other equations that go hand in hand with Maxwell's equations.

The Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

TIME-HARMONIC FIELDS:

So far, our time dependence of EM fields has been arbitrary. To be specific, we shall assume that the fields are *time harmonic.*

A **time-harmonic field** is one thai varies periodically or sinusoidally wiih time. Not only is sinusoidal analysis of practical value, it can be extended to most waveforms by Fourier transform techniques. Sinusoids are easily expressed in phasors, which are more convenient to work with.

Before applying phasors to EM fields, it is worthwhile to have a brief review of the concept of phasor.
A phasor z is a complex number that can be written as

$$z = x + jy = r \angle \phi$$

Herein lies the justification for using phasors; the time factor can be suppressed in our analysis of time-harmonic fields and inserted when necessary. Also note that in Table, the time factor *e'01'* has been assumed. It is equally possible to have assumed the time factor *e~ja"*, in which case we would need to replace every y in Table with —*j*.

TABLE: Time-Harmonic Maxwell's Equations Assuming Time Factor $e^{j\omega t}$

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

Electromagnectic Waves-1:

Our first application of Maxwell's equations will be in relation to electromagnetic wave propagation. The existence of EM waves, predicted by Maxwell's equations, was first investigated by Heinrich Hertz. After several calculations and experiments Hertz succeeded in generating and detecting radio waves, which are sometimes called Hertzian waves in his honor. In general, **waves** are means of transporting energy or information.

Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity; in traveling, they assume the properties of waves; and they radiate outward from a source, without benefit of any discernible physical vehicles.

Our major goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \gg \omega\varepsilon$)

Where w is the angular frequency of the wave. Case 3, for lossy dielectrics, is the most general case and will be considered first. Once this general case is solved, we simply derive other cases (1,2, and 4) from it as special cases by changing the values of *a*, e, and *ix.* However, before we consider wave motion in those different media, it is appropriate that we study the characteristics of waves in general. This is important for proper understanding of EM waves. Power considerations, reflection, and transmission between two different media will be discussed later.

Uniform plane wave propagation:

A clear understanding of EM wave propagation depends on a grasp of what waves are in general. **A wave** is a function of both space and time. Wave motion occurs when a disturbance at point A, at time *to,* is related to what happens at point *B,* at time *t* > *t0.* Partial differential equation of the second order. In one dimension, a scalar wave equation takes the form of

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0$$

where *u* is the *wave velocity.*
It can be solved by following procedure. Its solutions are of the form

$$E^- = f(z - ut)$$

$$E^+ = g(z + ut)$$

$$E = f(z - ut) + g(z + ut)$$

For the moment, let us consider the solution in above eq. Taking the imaginary part of this equation, we have

$$E = A \sin(\omega t - \beta z)$$

This is a sine wave chosen for simplicity; a cosine wave would have resulted had we taken the real part of above eq.

1. It is time harmonic because we assumed time dependence *ejo"* .
2. *A* is called the *amplitude* of the wave and has the same units as *E.*
3. *(ox* - /3z) is the *phase* (in radians) of the wave; it depends on time *t* and space variable *z.*
4. w is the *angular frequency* (in radians/second); 0 is the *phase constant* or *wave number* (in radians/meter).

Due to the variation of *E* with both time *t* and space variable *z,* we may plot £ as a function of *t* by keeping *z* constant and vice versa. The plots of *E(z, t* = constant) and *E(t, z* = constant) are shown in Figure 10.1(a) and (b), respectively. From Figure 10.1(a), we observe that the wave takes distance X to repeat itself and hence X is called the *wavelength* (in meters). From Figure , the wave takes time *T* to repeat itself; consequently *T* is known as the *period* (in seconds). Since it takes time *T* for the wave to travel distance X at the speed *u,* we expect

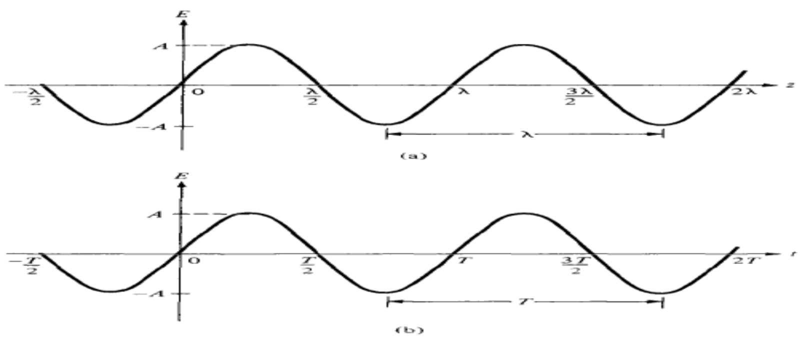
X = *uT*

But *T* = *l/f,* where/is the *frequency* (the number of cycles per second) of the wave in Hertz (Hz). Hence, Because of this fixed relationship between wavelength and frequency, one can identify the position of a radio station within its band by either the frequency or the wavelength. Usually the frequency is preferred. Also, because

$$\omega = 2\pi f$$

$$\beta = \frac{\omega}{u}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$



Plot of *E(z, t)*(b) with constant *z.*

(b)

• *A* sin(co/ - *&z):* (a) with constant *t*

$$\beta = \frac{2\pi}{\lambda}$$

Equation shows that for every wavelength of distance traveled, a wave undergoes a phase change of 2TT radians.We will now show that the wave represented by eq. is traveling with a velocity *u* in the +*z* direction. To do this, we consider a fixed point *P* on the wave,as the wave advances with time, point *P* moves along +*z* direction. Point *P* is a point of constant phase, therefore

$$\omega t - \beta z = \text{constant}$$

$$\frac{dz}{dt} = \frac{\omega}{\beta} = u$$

which is the same as eq shows that the wave travels with velocity *u* in the +*z* direction. Similarly, it can be shown that the wave *B* sin (cos + *(5z)* in eq. is traveling with velocity *u* in the — *z* direction.

In summary, we note the following:

1. A wave is a function of both time and space.
2. Though time / = 0 is arbitrarily selected as a reference for the wave, a wave is without beginning or end.
3. A negative sign in *(u>t ± /3z)* is associated with a wave propagating in the +*z* direction (forward travelling or positive-going wave) whereas a positive sign indicates that a wave is travelling in the —*z* direction (backward travelling or negative going wave).

Unit – 6

Objectives:

- To introduce the concepts of wave propagation and refraction of electromagnetic waves in different media.

Syllabus:

UNIT – VI: Electromagnetic Waves-II

Wave Propagation in lossless medium and conducting medium, Conductors and Dielectrics-Characterization. Polarization, Direction Cosines of normal to the plane of wave. Reflection and Refraction of Plane Waves – Normal and Oblique Incidences for Perfect Conductor and Perfect Dielectrics- Horizontal and Vertical Polarization, Poynting's theorem and Poynting's Vector.

Outcomes:

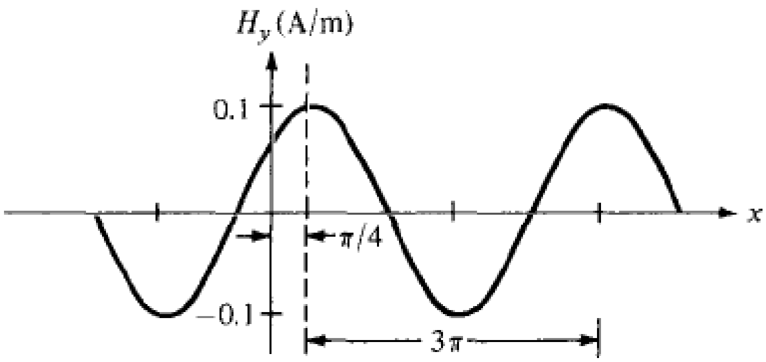
Students will be able

- Understand concept of wave propagation in different media.
- Reflection and refraction of plane waves.
- Understand the concept of polarization.
- Measure the power flowing through the given volume.

Our first application of Maxwell's equations will be in relation to electromagnetic wave propagation. The existence of EM waves, predicted by Maxwell's equations, was first investigated by Heinrich Hertz. In general, **waves** are means of transporting energy or information. Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity; in traveling, they assume the properties of waves; and they radiate outward
from a source, without benefit of any discernible physical vehicles

WAVE PROPAGATION IN LOSSY DIELECTRICS:

wave propagation in lossy dielectrics is a general case from which wave propagation in other types of media can be derived as special cases. Therefore, this section is foundational to the next three sections



A lossy **dielectric** is a medium in which an EM wave loses power as it propagates due to poor conduction.
In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with *a* ¥= 0, as distinct from a lossless dielectric (perfect or good dielectric) in which *a* = 0. Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free *(pv* = 0). Assuming and suppressing the time factor *ej'''*, Maxwell's equations

$$\nabla \cdot \mathbf{E}_s = 0$$

$$\nabla \cdot \mathbf{H}_s = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s$$

Taking the curl of both sides of eq. (10.13) gives

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu \, \nabla \times \mathbf{H}_s$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$\boxed{\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

and *y* is called the *propagation constant* (in per meter) of the medium. By a similar procedure, it can be shown that for the H field,

$$\nabla^2\mathbf{H}_s - \gamma^2\mathbf{H}_s = 0$$

The above Equations are known as homogeneous vector *Helmholtz 's equations* or simply vector *wave equations.*

$$\gamma = \alpha + j\beta$$

We obtain *a* and beeta from eqs

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

A sketch of |E| at times *t* = 0 and *t* = *At* is portrayed , where it is evident that E has only an x-component and it is traveling along the +z-direction. Having obtained E(z, *t),* we obtain H(z, *t)* either by taking similar steps to solve or by using eq. in conjunction with Maxwell's equations , We will eventually arrive at

$$\mathbf{H}(z, t) = \text{Re}\,(H_o e^{-\alpha z} e^{j(\omega t - \beta z)}\,\mathbf{a}_y)$$

where

$$H_o = \frac{E_o}{\eta}$$

And eeta is a complex quantity known as the *intrinsic impedance* (in ohms) of the medium. It can be shown by following the steps taken in

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle\theta_\eta = |\eta| e^{j\theta_\eta}$$

E and H are out of phase by 0, at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, E leads H (or H lags E) by *6V*. Finally, we notice that the ratio of the magnitude of the conduction current density J to that of the displacement current density *Jd* in a lossy medium **is**

$$\frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega\varepsilon \mathbf{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

$$\boxed{\tan\theta = \frac{\sigma}{\omega\varepsilon}}$$

PLANE WAVES IN LOSSLESS DIELECTRICS:

In a lossless dielectric, *a* <$C we. It is a special case of that

$$\sigma \simeq 0, \qquad \varepsilon = \varepsilon_0\varepsilon_r, \qquad \mu = \mu_0\mu_r$$

Substituting these in general case.

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta}$$

Also

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0^\circ$$

and thus **E** and **H** are in time phase with each other.

PLANE WAVES IN FREE SPACE:

$$\sigma = 0, \qquad \varepsilon = \varepsilon_0, \qquad \mu = \mu_0$$

This may also be regarded as a special case, Thus we simply replace e by eo and *k* by /xo in eq Either way, we obtain

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c}$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta}$$

where *c* =3 X 108 m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters *dv* = 0 and *V* = ^oi where rjo is called the *intrinsic impedance of free space* and is given by

$$\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\ \Omega$$

PLANE WAVES IN GOOD CONDUCTORS:

This is another special case of that considered , A perfect, or good conductor,

$$\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r$$

Hence the charecteristics

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta}$$

Also,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ$$

Skin depth:

Therefore, as E (or H) wave travels in a conducting medium, its amplitude is attenuated by the factor *e~az.* The distance <5, through which the wave amplitude decreases by a factor *e~l* (about 37%) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

or

$$\delta = \frac{1}{\alpha}$$

The **skin depth** is a measure of the depth to which an EM wave can penetrate the Medium.

Normal and oblique Incidence:

Incident Wave:

(E,, H,) is traveling along +az in medium 1. If we suppress the time factor $e^{j\phi t}$ and assume that

$$\mathbf{E}_{is}(z) = E_{io}e^{-\gamma_1 z}\,\mathbf{a}_x$$

$$\mathbf{H}_{is}(z) = H_{io}e^{-\gamma_1 z}\,\mathbf{a}_y = \frac{E_{io}}{\eta_1}e^{-\gamma_1 z}\,\mathbf{a}_y$$

Reflected Wave: (Er, Hr) is traveling along +*az* in medium 1. If Ers(z) = *Er0*$e^{\gamma z}$$\boldsymbol{a_x}$

$$\mathbf{H}_{rs}(z) = H_{ro}\,e^{\gamma_1 z}(-\mathbf{a}_y) = -\frac{E_{ro}}{\eta_1}e^{\gamma_1 z}\,\mathbf{a}_y$$

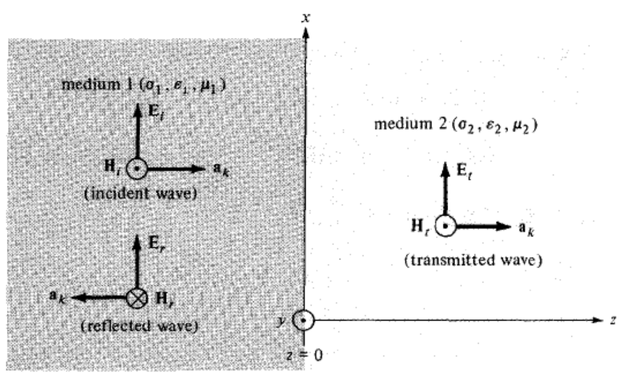


Fig.:A Plane wave Incident normally on an Interface between two different media

Transmitted Wave:

(E_t, Ht) is traveling along +*az* in medium 2. If

$$\mathbf{E}_{ts}(z) = E_{to}\,e^{-\gamma_2 z}\,\mathbf{a}_x$$

$$\mathbf{H}_{ts}(z) = H_{to}\,e^{-\gamma_2 z}\,\mathbf{a}_y = \frac{E_{to}}{\eta_2}e^{-\gamma_2 z}\,\mathbf{a}_y$$

Then reflection coefficient

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

And

$$\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

By solving the electric component of the wave is

$$\mathbf{E}_1 = 2E_{io} \sin \beta_1 z \sin \omega t \, \mathbf{a}_x$$

Similarly the magnetic field component of the wave is

$$\mathbf{H}_1 = \frac{2E_{io}}{\eta_1} \cos \beta_1 z \cos \omega t \, \mathbf{a}_y$$

REFLECTION OF A PLANE WAVE AT OBLIQUE INCIDENCE:

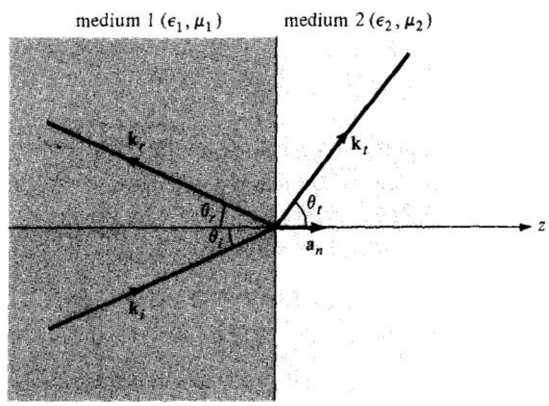


Fig.:Oblique Incidence of a plane wave

$$n_1 \sin \theta_i = n_2 \sin \theta_t$$

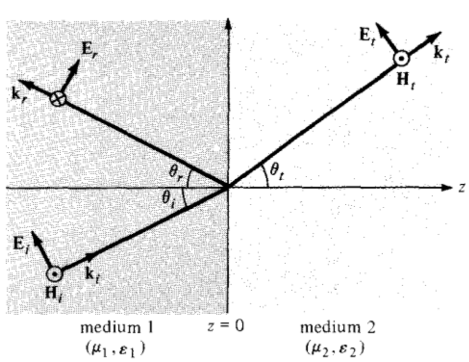


Fig.: Oblique incidence with E parallel to the plane of incidence.

$$\Gamma_{\|} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$\tau_{\|} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

The above equations are called the Fresnel Equations.

Brewsters Angle:

$$\sin^2 \theta_{B_{\|}} = \frac{1 - \mu_2\varepsilon_1/\mu_1\varepsilon_2}{1 - (\varepsilon_1/\varepsilon_2)^2}$$

Perpendicular Polarization:

E field is perpendicular to the plane of Incidence.

$$\mathbf{E}_{is} = E_{io} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}\, \mathbf{a}_y$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1}(-\cos\theta_i\, \mathbf{a}_x + \sin\theta_i\, \mathbf{a}_z)\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{E}_{rs} = E_{ro} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\, \mathbf{a}_y$$

$$\mathbf{H}_{rs} = \frac{E_{ro}}{\eta_1}(\cos\theta_r\, \mathbf{a}_x + \sin\theta_r\, \mathbf{a}_z)\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

While the transmitted field inmedium 2 is given by

$$\mathbf{E}_{ts} = E_{to} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\, \mathbf{a}_y$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta_2}(-\cos\theta_t\, \mathbf{a}_x + \sin\theta_t\, \mathbf{a}_z)\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Then on solving

$$\Gamma_{\perp} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\tau_{\perp} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

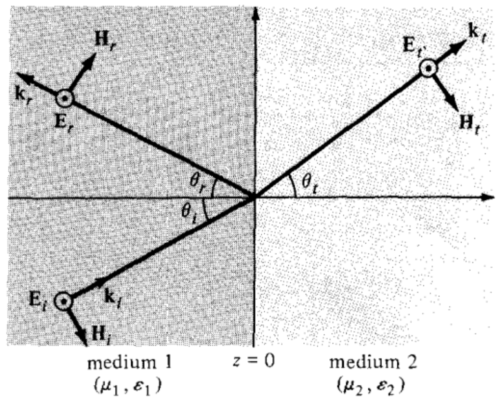


Fig.:Oblique incidence with E perpendicular to the plane of incidence

POWER AND THE POYNTING VECTOR:

Energy can be transmitted from one point to another point by means of EM Waves. The rate of such energy transportation can be obtained from Maxwell's equations

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Dotting both sides of eq. with **E** gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\varepsilon t}$$

But for any vector fields **A** and **B**

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

Applying this vector identity to eq.

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{H} \times \mathbf{E}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\mu}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H})$$

and thus eq. becomes

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral of both sides,

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right] dv - \int_v \sigma E^2\, dv$$

Applying the divergence theorem to the left-hand side gives

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right] dv - \int_v \sigma E^2\, dv$$

↓ Total power leaving the volume = ↓ Rate of decrease in energy stored in electric and magnetic fields − ↓ Ohmic power dissipated

The above equation is referred to as Poynting theorem . The pointing vector is denoted by

$$\mathcal{P} = \mathbf{E} \times \mathbf{H}$$